Graphs and Combinatorics 2, 21–29 (1986)

Graphs and Combinatorics

Uniqueness and Nonexistence of Some Graphs Related to M_{22}

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Abstract. There is a unique distance regular graph with intersection array $i(7,6,4,4;1,1,1,6)$; it has 330 vertices, and its automorphism group $M_{22}.2$ acts distance transitively. It does not have an antipodal 2-cover, but it has a unique antipodal 3-cover, and this latter graph has automorphism group $3.M_{22}.2$ acting distance transitively. As a side result we show uniqueness of the strongly regular graph with parameters $(v,k,\lambda,\mu) = (231,30,9,3)$ under the assumption that it is a gamma space with lines of size 3.

1. Uniqueness of the Cameron Graph

There exists a strongly regular graph (sometimes called the Cameron graph) on 231 vertices with full automorphism group $M_{22}.2$ constructed by taking as vertices the unordered pairs from a 22-set and joining two pairs whenever they are disjoint and their union is contained in a block of a (fixed) Steiner system $S(3,6,22)$ on this 22-set. (For undefined terminology, see e.g. Cameron & van Lint [3].) This graph becomes the collinearity graph of a partial linear space with lines of size 3 if one takes as lines the triples of pairwise disjoint pairs whose union is a block of the Steiner system. This partial linear space is a gamma space, that is, given a line L and a point x outside, then x is collinear with zero, one or all points of L. The next theorem shows that this property characterizes our graph.

Theorem 1. *Let $(X, \mathscr{L})$ be a gamma space with lines of size 3 such that its collinearity graph Γ is strongly regular with parameters $(v,k,\lambda,\mu) = (231,30,9,3)$. Then Γ is isomorphic to the Cameron graph described above.*

(Here, following common practice but unlike [3], v denotes the number of vertices, k the valency, λ the number of common neighbours of two adjacent vertices and μ the number of common neighbours of two nonadjacent vertices of a graph Γ.)

Proof. Write $\Gamma(x)$ for the set of neighbours of a vertex x; $\mu(x,y) = \Gamma(x) \cap \Gamma(y)$ for the set of common neighbours of two nonadjacent vertices x and y. The graph induced by Γ on $\mu(x,y)$ is called a μ-graph.

(1) Each μ-graph is either a 3-coclique or a line.
(For: each $\{x\} \cup \Gamma(x)$ is a subspace of $(X, \mathscr{L})$, and the intersection of subspaces is again a subspace.)

(2) Each vertex is in ten 7-cliques and each line in two 7-cliques; each 7-clique is a subspace isomorphic to the Fano plane.
(For: maximal cliques are subspaces and have at most 11 vertices (e.g. because $\lambda = 9$), and at least 7 vertices; but no STS (11) exists, so there are two maximal cliques on each line and each has 7 vertices.)

The m-clique extension of a graph is obtained by replacing each vertex x by an m-clique C_x, and joining each vertex of C_x to each vertex of C_y whenever $x \sim y$.

(3) For each vertex x, $\Gamma(x)$ is isomorphic to the 2-clique extension of the line graph of the Petersen graph.
(For: form a graph Δ with the Fano planes on x as vertices and the lines on x as edges and (reverse) inclusion as incidence. Then Δ has 10 vertices, valency 3, no triangles and no quadrangles (otherwise Γ would have $\mu \geq 5$), so Δ is the Petersen graph.)

Note that the line graph of the Petersen graph is an antipodal 3-cover of the complete graph K_5 so that we have a concept of antipodal concurrent lines.

(4) Each line L is contained in a unique subspace isomorphic to the $GQ(2,2)$ generalised quadrangle (For: let $L = \{x, y, z\}$ and let M, N be the two lines on x antipodal to L, say $M = \{x, u, v\}$. Let p be a common neighbour of u and y distinct from x. Since M is antipodal to L we have that $\mu(u, y)$ is a 3-coclique, so $p \nsim x$ and $\mu(p, x)$ is a 3-coclique, so py is antipodal to L and pu is antipodal to M. It follows that the 8 common neighbours of a point of $L \backslash \{x\}$ and a point of $M \backslash \{x\}$ lie on the 8 lines not on x antipodal to L or M, and we find two 3×3 grids having $L \cup M$ in common. But these same 8 points are also joined to $N \backslash \{x\}$ by the 4 lines not on x antipodal to N, and the 15 points and 15 lines we have found form a $GQ(2,2)$. Uniqueness follows since in a $GQ(2,2)$ all μ-graphs are 3-cocliques so that any two intersecting lines are antipodal and the whole construction was forced.)

Let us call a $GQ(2,2)$ subgraph (subgeometry) a *quad*.

(5) There are 77 quads, 5 on each vertex, 1 on each line, and any two have at most one vertex in common. Two nonadjacent vertices x, y are in a quad if and only if $\mu(x, y)$ is a 3-coclique. Quads are geodetically closed.
(For: if $x \nsim y$ and $\mu(x, y)$ is a 3-coclique and p is a common neighbour of x and y then the lines px and py are antipodes, and y is in the unique quad containing px.)

We shall write $Q(L)$ and $Q(x, y)$ for the unique quad on the line L or on the nonadjacent vertices x, y (this notation implying that $\mu(x, y)$ is a 3-coclique).

(6) Let Q be a quad and $x \notin Q$. Then $\Gamma(x) \cap Q$ is either empty or a line. If we write $\Gamma_i(Q) = \{y \mid d(y, Q) = i\}$ then $|\Gamma_0(Q)| = 15$, $|\Gamma_1(Q)| = 120$, $|\Gamma_2(Q)| = 96$.
(For: let L be a line on x meeting Q in y, then L is in a Fano plane together with one of the three lines on y in Q.)

(7) If Q, Q' are two quads, and $Q \cap Q' = \{z\}$ then the 8 nonneighbours of z in Q' are in $\Gamma_2(Q)$. There are 60 quads meeting Q in a single point, 5 on each point of $\Gamma_2(Q)$, so there are 16 quads disjoint from Q and these are entirely contained within $\Gamma_1(Q)$.
(For: let $x \in Q'$, $y \in Q$, $x \sim y$, $x \nsim z$ then $\Gamma(x) \cap Q$ is a line L on y. This line does not contain z, so z has a neighbour on it and we may assume $z \sim y$. But now $x \sim y \sim z$ and Q' is geodetically closed, so $y \in Q'$, contradiction.)

(8) There are no three pairwise disjoint quads.

(For: suppose Q_1, Q_2, Q_3 are pairwise disjoint, and define γ_{ij}: $Q_i \to Q_j^*$ by $\gamma_{ij}(x) = \Gamma(x) \cap Q_j$, where Q^* denotes the generalized quadrangle dual to Q. Then $\psi = \gamma_{32}^{-1} \circ \gamma_{12}$ is an isomorphism from Q_1 onto Q_3. If $x \in Q_1$ and $x \nsim \psi(x)$ then $\mu(x, \psi(x))$ is the line $\gamma_{12}(x)$, but $\psi(x)$ also has a neighbour on the line $\gamma_{13}(x)$, contradiction. Thus $x \sim \psi(x)$ for each $x \in Q$, and $\gamma_{13} \circ \psi^{-1}$ is a polarity of Q_3 where all points are absolute. But $GQ(2,2)$ has no such polarity, contradiction.)

(9) For a graph Δ with the quads as vertices, two quads being adjacent whenever they are disjoint. Then Δ is the unique strongly regular graph with parameters $(v, k, \lambda, \mu) = (77, 16, 0, 4)$ and is isomorphic with the graph that has the blocks of $S(3, 6, 22)$ as vertices and pairs of disjoint blocks as edges.
(For: we have seen v, k, λ and $\mu = 4$ is easily checked. Now the result follows from Brouwer [2].)

Now we might continue describing Γ in terms of Δ, exploiting detailed knowledge of Δ. Instead I'll choose another way, showing the rank 4 structure of Γ.

(10) Γ carries a 3-class association scheme with $(x, y) \in R_0$ iff $x = y$, $(x, y) \in R_1$, iff $x \sim y$, $(x, y) \in R_2$ iff $x \nsim y$ and $\mu(x, y)$ is a line $(x, y) \in R_3$ iff $x \nsim y$ and $\mu(x, y)$ is a 3-coclique. The parameters are $(p^i_{0j}) = I$,

$$(p^i_{1j})_{ij} = \begin{pmatrix} 0 & 30 & 0 & 0 \\ 1 & 9 & 16 & 4 \\ 0 & 3 & 21 & 6 \\ 0 & 3 & 24 & 3 \end{pmatrix}, \quad (p^i_{2j})_{ij} = \begin{pmatrix} 0 & 0 & 160 & 0 \\ 0 & 16 & 112 & 32 \\ 1 & 21 & 108 & 30 \\ 0 & 24 & 120 & 16 \end{pmatrix},$$

$$(p^i_{3j}) = \begin{pmatrix} 0 & 0 & 0 & 40 \\ 0 & 4 & 32 & 4 \\ 0 & 6 & 30 & 4 \\ 1 & 3 & 16 & 20 \end{pmatrix}.$$

(For:

a) $p^1_{33} = 4$: If $(x, y) \in R_1$, (x, z), $(y, z) \in R_3$ then by (7) x, y, z are all in one quad, the unique quad on xy, and in this quad there are 4 points nonadjacent to x and y.

b) $P^2_{33} = 4$: Suppose $\mu(x, y) = L$. There are two quads on x disjoint from L, and if Q is such a quad then $y \notin Q$ and $\Gamma(y) \cap Q = \varnothing$ (otherwise $\Gamma(y) \cap Q$ would be a line and $\mu(x, y)$ would contain at least 4 points); y is in 5 quads and each meets Q in a single point. These 5 points form an ovoid, as follows from a) and hence 3 of them are adjacent to x. Remain 2 possibilities for z with $(x, z) \in R_3$ on each Q, so $p^2_{33} = 4$.

c) $p^1_{13} = 4$: Suppose $x \sim y \sim z$, $(x, z) \in R_3$. Then $Q(x, z)$ is the unique quad on xy and z is one of the 4 neighbours of y not on xy in this quad.

d) $p^3_{13} = 3$: Suppose $x \sim z$, (x, y), $(y, z) \in R_3$. Then x, y, z are all in one quad $Q(x, y)$ and z is one of the 3 neighbours of x nonadjacent to y in this quad.

e) $p^2_{13} = 6$: Suppose $x \sim z$, $(x, y) \in R_2$, $(y, z) \in R_3$. Let $Q = Q(y, z)$. By b) the line $L = \mu(x, y)$ meets Q, in a point p, say. Now $\Gamma(x) \cap Q$ is the line pz, and Q is the unique quad containing the line py. For p there are 3 choices on L and in each case we find two possibilities for z.

f) $p^3_{33} = 20$: Suppose $(x, y) \in R_3$, $Q = Q(x, y)$. Inside Q there are 4 points non-adjacent to x and y. Any other point z with (x, z), $(y, z) \in R_3$ must be in $\Gamma_2(Q)$.

If $z \in \Gamma_2(Q)$ then the five quads on z meet Q in five points forming an oval O in Q. Now Q has 6 ovals, and if O, O' are any two ovals then there are precisely 32 points z determining either O or O' (for: $O \cap O' = \{p\}$ and there are 32 points nonadjacent to p in the four quads distinct from Q on p); it follows that any given oval (and in particular the one containing x, y) is determined by 16 points z.

Thus $p_{33}^3 = 4 + 16 = 20$.

All other p_{jk}^i are determined by these (and the parameters of Γ as a strongly regular graph).)

(11) (x, R_3) is isomorphic with the triangular graph $\binom{22}{2}$.

(For: it has the right parameters by (10), and uniqueness follows by Connor [4].)

Let us identify the quads in this triangular graph.

Lemma. *Let Δ be a triangular graph $\binom{n}{2}$ and T a noncomplete subgraph isomorphic to $\binom{m}{2}$. If Δ is labelled with $\binom{Y}{2}$ for some n-set Y then this labelling induces a labelling with $\binom{Z}{2}$ on T, where Z is some m-subset of Y. (In other words, there are only canonical ways to embed noncomplete triangular subgraphs.)*

Proof. Let x, y be vertices of T labelled with ab and ac respectively. We prove that some vertex of T is labelled with bc. Choose $z \in T$, $z \sim y$, $z \nsim x$. Then z is labelled with cd, say. Now $\mu(x, z)$ is a 4-circuit, so there are two vertices, $u, v \in T$ adjacent to each of x, y, z. But these must be labelled ad and bc. □

(This Lemma reminds me of recent work by J.I. Hall on Kneser graphs – probably it is a special case of some of his results.)

(12) Γ is the graph with as vertices the pairs from a set of 22 symbols, where two pairs are adjacent whenever they are disjoint and their union is contained in a block of a $S(3, 6, 22)$ design on the set of symbols.

(For: the collinearity graph of a quad $GQ(2, 2)$ is the complement of the triangular graph $\binom{6}{2}$, so by the Lemma and (11) we can label X with the pairs from a set Σ of 22 symbols and the quads correspond to certain 6-subsets of Σ. Each triple in Σ determines a unique quad, so these 6-sets form a Steiner system $S(3, 6, 22)$ on Σ. If two pairs are adjacent in Γ then they are nonadjacent in (X, R_3), i.e. disjoint, and they are contained in a quad.) □

Remark. The association scheme described in (10) corresponds to the group action, i.e., M_{22} acts rank 4 on X with suborbits $1 + 30 + 160 + 40$.

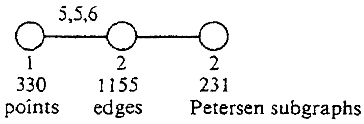
2. Uniqueness of a Graph on 330 Vertices

Define a graph Γ with as vertices the 330 blocks of the Steiner system $S(5, 8, 24)$ missing two fixed symbols, where two blocks are adjacent whenever they are disjoint.

We have the following correspondence between graph distance and size of intersection:

$\|B \cap B'\|$	$d(B, B')$	
8	0	
0	1	
2	3	
4	2	if the sextet determined by B and B' has both fixed symbols in the same tetrad,
	4	otherwise.

It is easy to check Γ is distance regular (in fact, distance transitive) with intersection array $i(7, 6, 4, 4; 1, 1, 1, 6)$. Γ has full group of automorphisms $M_{22}.2$. The vertices, edges and Petersen subgraphs of Γ form a geometry with Buekenhout diagram

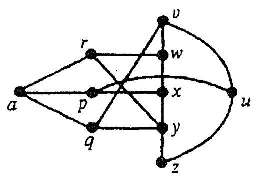


Our aim here is to prove uniqueness of Γ from its parameters. Let us start with a lemma producing the Petersen subgraphs.

Lemma. *Let Γ be a graph with $\mu = c_3 = 1$ and $\lambda = 0, a_2 = 2$. Then any two vertices at distance two in Γ determine a unique induced Petersen graph.*

(For notation, see Biggs [1]; we do not suppose that Γ is distance regular.)

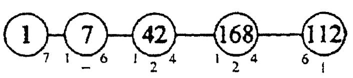
Proof.

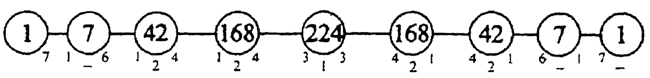


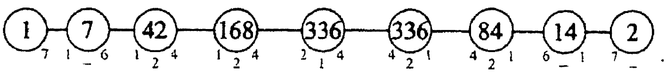
Let $d(a, x) = 2$, with $a \sim p \sim x$. Let x have heighbours w, y in $\Gamma_2(a)$, with $a \sim r \sim w$, $a \sim q \sim y$. Since Γ has girth 5, all points mentioned are distinct. Since w has two neighbours (r and x) in $\Gamma_2(q)$ we have $d(w, q) = 2$, sy $q \sim v \sim w$, and similarly $r \sim z \sim y$. Again $d(v, z) = 2$ (not $v \sim z$, since $v \sim z \sim r \sim w \sim v$ would be a 4-circuit, and not $d(v, z) = 3$ since $\{w, q\} \subset \Gamma(v) \cap \Gamma_2(z)$) so $v \sim u \sim z$ for some vertex u. We must have $d(a, u) = 2$ and $d(p, v) = 2$ so $p \sim u$, completing our Petersen graph. □

(We find that there are $k(k-1)/6$ Petersen graphs on a vertex and $vk(k-1)/60$ Petersen graphs altogether, so these numbers must be integers for a graph Γ satisfying the hypotheses of the Lemma.)

This Lemma applies to our graph on 330 vertices as well as to its antipodal 2-covers and 3-covers (these pass all known existence criteria). The three distance distribution diagrams are

$v = 330$ 

$v = 660$ 

$v = 990$ 

In these cases we have $k = 7$, each vertex x is in 7 Petersen graphs and the triples induced by these on $\Gamma(x)$ must form the Fano plane; in particular, two Petersen graphs on a vertex x have an edge in common.

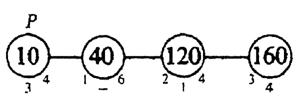
Proposition. *There is no distance regular graph on* 660 *vertices with intersection array* $i(7, 6, 4, 4, 3, 1, 1, 1; 1, 1, 1, 3, 4, 4, 6, 7)$.

Proof. Let Γ be such a graph. Choose vertices x_0, a, b, x_8 with $d(x_0, x_8) = 8$, $a \sim b$, $\{a, b\} \subset \Gamma_4(x_0) \cap \Gamma_4(x_8)$. Since $c_4 = 3$ is odd, there must be a Petersen graph P on the edge ab meeting both $\Gamma_3(x_0) \cap \Gamma(a)$ and $\Gamma_3(x_8) \cap \Gamma(a)$, say $x_3 \in P \cap \Gamma(a) \cap \Gamma_3(x_0)$, $x_5 \in P \cap \Gamma(a) \cap \Gamma_3(x_8)$. Let $x_0 \sim x_1 \sim x_2 \sim x_3 \sim a \sim x_5 \sim x_6 \sim x_7 \sim x_8$ be a geodesic. P must have an edge in common with the Petersen graph determined by x_1 and x_3, and this edge lies in $\Gamma_3(x_0)$; similarly, P must have an edge in $\Gamma_5(x_0)$ – but one easily sees that this is impossible since $a_4 = 1$. □

Theorem 2. *There is a unique distance regular graph* Γ *on* 330 *vertices with intersection array* $i(7, 6, 4, 4; 1, 1, 1, 6)$.

Proof. We first show that the graph Δ with as vertices the Petersen subgraphs of Γ where two Petersen graphs are adjacent when they meet, is isomorphic to the Cameron graph. First of all Δ has 231 vertices and valency 30.

Claim. The distance distribution around a Petersen graph P is



Indeed: P is geodetically closed so any point in $\Gamma_1(P)$ has a unique neighbour in P. Also, if $x, y \in P$ with $d(x, y) = 2$ then the two neighbours of y at distance two to x are in P and it follows that $\Gamma_1(P)$ is a coclique. If $z \in \Gamma_2(P)$ then at most 7 points of P can be in $\Gamma_2(z)$, so there are points of P in $\Gamma_3(z)$. Now since $c_3 = 1$ we must have that $\Gamma_2(z) \cap P$ is geodetically closed and hence is an edge. If P' is the unique Petersen graph on z meeting P then P' contains an edge on z in $\Gamma_2(P)$, so z has at most 4 neighbours in $\Gamma_3(P)$. The maximum possible distance to P is 3 since Γ has diameter 4 and $a_4 = 1 < 3$. If some point $u \in \Gamma_3(P)$ had at most two neighbours in $\Gamma_2(P)$ then $|P \cap \Gamma_3(u)| \leq 4$, and removing at most two edges from P we are left with a graph where each vertex has degree at most one – impossible. Thus the number of edges between $\Gamma_2(P)$ and $\Gamma_3(P)$ is both at most and at least $480 = 4.120 = 3.160$ and we have equality everywhere, proving the claim.

Let us compute λ. If P, P' and P'' are three Petersen graphs that have pairwise nonempty intersection then by the previous $P \cap P' \cap P''$ is nonempty. Let $P \cap P' = \{u, v\}$, then there is one more Petersen graph on $\{u, v\}$, and 4 others on u and on v, so that $\lambda = 1 + 4 + 4 = 9$.

Next look at μ. There are two possibilities (as was to be expected, since the Cameron graph is rank 4, not rank 3): (i) P' meets $\Gamma_1(P)$, and (ii) $d(P, P') > 1$.

In the first case we see from "$a_2(P) = 1$" that any P'' meeting both P and P' must contain the (unique, since $c_3 = 1$) edge joining P and P' so that P and P' have three common neighbours.

In the second case we see from the distance distribution diagram around P and the fact that any two Petersen graphs on a point have an edge in common, that P' contains 3 edges in $\Gamma_2(P)$.

(Indeed, if $u \in \Gamma_3(P)$ then u is in 6 Petersen graphs meeting $\Gamma_1(P)$, two on each edge uv with $v \in \Gamma_2(P)$, so u is in a unique Petersen graph P' not meeting $\Gamma_1(P)$, and P' contains the three neighbours of u in $\Gamma_2(P)$ so that $P' \cap \Gamma_3(P)$ is a coclique. But the only way to split a Petersen graph into a coclique and a graph where each vertex has degree (at most) one is as $\bar{K}_4 + 3K_2$.)

Thus $\mu = 3$, and by Theorem 1 the graph Δ is isomorphic to the Cameron graph. (Clearly the 3-lines of Δ are the triples of Petersen graphs on a given edge, and the computation of λ also proved the Gamma space property.)

[This gives us a 22-set Σ and a Steiner system $S(3, 6, 22)$ on Σ and a labelling of Δ with $\binom{\Sigma}{2}$ such that Petersen graphs at distance 2 correspond to intersecting pairs and intersecting Petersen graphs correspond to disjoint pairs contained in a block of the Steiner system. We want to let the vertices of Γ correspond to 8-subsets of Σ. This is done as follows:

Given a vertex x, it is in 7 Petersen graphs labelled with 7 pairwise disjoint pairs of symbols. Label x with the set of $22 - 2.7 = 8$ remaining symbols. We shall however not use this labelling.]

Each vertex x determines a 7-clique in Δ, and we find 330 7-cliques in Δ in this way; but Δ has only 330 7-cliques, 10 on each vertex of Δ, so we indentify Γ as the graph with as vertices the Fano planes in Δ, where two Fano planes are adjacent when they have a line in common. This shows that Γ is uniquely determined. □

Remarks. The Petersen subgraphs arise as follows:

Let α, β be the two fixed symbols chosen in the symbol set $\Sigma \cup \{\alpha, \beta\}$ in order to define Γ. Any sextet such that α and β lie in the same tetrad T of the sextet has 5 remaining tetrads, and the union of the any two of these is a block of $S(5, 8, 24)$, giving 10 blocks altogether, and these 10 blocks induce a Petersen subgraph in Γ. The pair this Petersen graph is labelled with is $T \backslash \{\alpha, \beta\}$, showing that the labelling proposed above is the correct one.

The suborbit lengths $(1 + 7 + 42 + 168 + 112)$ were given incorrectly by Fischer & McKay [5] but are stated correctly in Ivanov, Klin & Faradjev

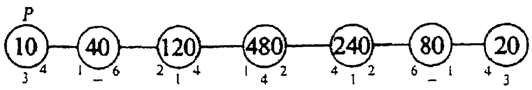
[6]. The full automorphism group of Γ is $M_{22}.2$, but already M_{22} acts distance transitively.

3. Uniqueness of a Graph on 990 Vertices

Recently, existence of a distance transitive graph on 990 vertices with intersection array $i(7, 6, 4, 4, 4, 1, 1, 1; 1, 1, 1, 2, 4, 4, 6, 7)$ was shown by Ivanov, Ivanov & Faradjev [7]. Its full automorphism group is $3.M_{22}.2$; it is already distance transitive under $3.M_{22}$. This graph is interesting for several reasons; for instance, it provides an example of a distance regular graph where the sequence $(a_j)_{0 \le j \le d}$ is not unimodal.

Theorem 3. *There is a unique distance regular graph* $\tilde{\Gamma}$ *on* 990 *vertices with intersection array* $i(7, 6, 4, 4, 4, 1, 1, 1; 1, 1, 1, 2, 4, 4, 6, 7)$.

Proof. As before we find Petersen graphs; the distance distribution around a Petersen graph P is



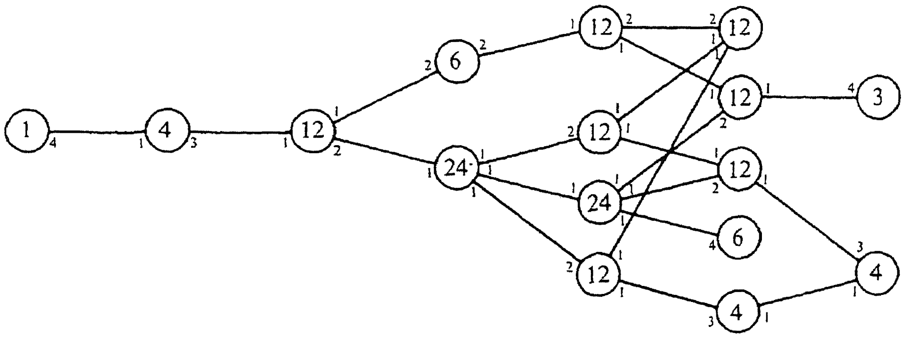
Let $\tilde{\Gamma}$ be an antipodal 3-cover of Γ. Pick a Petersen graph P in Γ and a vertex $x \in \Gamma_3(P)$. Then $\Gamma_3(x) \cap P \simeq 3K_2$. Since P has only five subgraphs isomorphic to $3K_2$ we see that $\Gamma_3(P)$ is a 32-cover of the complete graph K_5. Put $\Delta = \Gamma_3(P)$ and consider the inverse images of P, x and Δ in $\tilde{\Gamma}$. Above P we see three Petersen graphs $\tilde{P}_1, \tilde{P}_2, \tilde{P}_3$ at mutual distance 6. If $\tilde{x}$ is one of the three vertices above x then $\tilde{\Gamma}_3(\tilde{x}) \cap \tilde{P}_j$ is a single edge ($j = 1, 2, 3$) so that we find a labelling of the three edges in $\Gamma_3(x) \cap P$ with $\{1, 2, 3\}$. If $\tilde{x} \sim \tilde{y} \in \tilde{\Gamma}_3(\tilde{P}_j)$ then the labelling of the three edges in $\Gamma_3(y) \cap P$ determined by $\tilde{y}$ is given by the requirement that each edge in $\Gamma_3(y) \cap P$ has the same label as the edge in $\Gamma_3(x) \cap P$ it meets.

If $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ are the three vertices above x then these determine three labellings of the three edges in $\Gamma_3(x) \cap P$ that are cyclic shifts of each other (since for $i \neq j$ we have $d(\tilde{x}_i, \tilde{x}_j) = 8$ so $\tilde{x}_i$ and $\tilde{x}_j$ cannot both have distance 3 to the same vertex of some $\tilde{P}_h$).

Now Δ is connected, so identifying the vertex set of $\tilde{\Delta}$ with $\Delta \times \mathbb{Z}_3$ all adjacencies in $\tilde{\Delta}$ are determined and clearly this determines $\tilde{\Gamma}$. Thus there is at most one possibility for $\tilde{\Gamma}$ and by the result of Ivanov, Ivanov & Faradjev there is a unique $\tilde{\Gamma}$. □

Remark. I have not determined whether $\tilde{\Delta}$ is the union of three copies of Δ or is connected, since that is unimportant for the above argument.

The distance distribution diagram for Δ follows. (We have $v = 160$, $k = (1, 4, 12, 30, 60, 46, 7)$.)



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